Solving this system of equations gives $A = 5$, $B = 2$, and $C = 1$; therefore, the original integral can be written as

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx = \int \frac{5}{x-2}\,dx + \int \frac{2x+1}{x^2-2x+3}\,dx.$$

Let's work on the second (more difficult) integral. The substitution $u = x^2 - 2x + 3$ would work if $du = (2x-2)\,dx$ appeared in the numerator. For this reason, we write the numerator as $2x + 1 = (2x - 2) + 3$ and split the integral:

$$\int \frac{2x+1}{x^2-2x+3}\,dx = \int \frac{2x-2}{x^2-2x+3}\,dx + \int \frac{3}{x^2-2x+3}\,dx$$

Assembling all the pieces, we have

$$\begin{aligned}
&\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx \\
&= \int \frac{5}{x-2}\,dx + \int \frac{2x-2}{\underbrace{x^2-2x+3}_{\text{let } u = x^2-2x+3}}\,dx + \int \frac{3}{\underbrace{x^2-2x+3}_{(x-1)^2+2}}\,dx \\
&= 5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C \qquad \text{Integrate.} \\
&= \ln|(x-2)^5(x^2-2x+3)| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C \qquad \text{Property of logarithms}
\end{aligned}$$

To evaluate the last integral $\int \frac{3\,dx}{x^2-2x+3}$, we completed the square in the denominator and used the substitution $u = x - 1$ to produce $\int \frac{3\,du}{u^2+2}$, which is a standard form.

Related Exercises 30–36 ◂

Final Note The preceding discussion of partial fraction decomposition assumes that $f(x) = p(x)/q(x)$ is a proper rational function. If this is not the case and we are faced with an improper rational function f, we divide the denominator into the numerator and express f in two parts. One part will be a polynomial, and the other will be a proper rational function. For example, given the function

$$f(x) = \frac{2x^3 + 11x^2 + 28x + 33}{x^2 - x + 6}$$

we perform long division:

$$\begin{array}{r}
2x + 13 \\
x^2 - x + 6 \,\big)\, 2x^3 + 11x^2 + 28x + 33 \\
2x^3 - 2x^2 + 12x \\
\hline
13x^2 + 16x + 33 \\
13x^2 - 13x + 78 \\
\hline
29x - 45
\end{array}$$

It follows that

$$f(x) = \underbrace{2x + 13}_{\substack{\text{polynomial}\\ \text{easy to}\\ \text{integrate}}} + \underbrace{\frac{29x - 45}{x^2 - x + 6}}_{\substack{\text{apply partial fraction}\\ \text{decomposition}}}.$$

The first piece is easily integrated, and the second piece now qualifies for the methods described in this section.

SUMMARY Partial Fraction Decompositions

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor** A factor $x - r$ in the denominator requires the partial fraction $\frac{A}{x-r}$.
2. **Repeated linear factor** A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$
3. **Simple irreducible quadratic factor** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax + B}{ax^2 + bx + c}.$$
4. **Repeated irreducible quadratic factor** (See Exercises 67–70.) An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}.$$

SECTION 7.4 EXERCISES

Review Questions

1. What kinds of functions can be integrated using partial fraction decomposition?
2. Give an example of each of the following.
   a. A simple linear factor
   b. A repeated linear factor
   c. A simple irreducible quadratic factor
   d. A repeated irreducible quadratic factor
3. What term(s) should appear in the partial fraction decomposition of a proper rational function with each of the following?
   a. A factor of $x - 3$ in the denominator
   b. A factor of $(x - 4)^3$ in the denominator
   c. A factor of $x^2 + 2x + 6$ in the denominator
4. What is the first step in integrating $\frac{x^2 + 2x - 3}{x + 1}$?

Basic Skills

5–8. Setting up partial fraction decomposition *Give the appropriate form of the partial fraction decomposition for the following functions.*

5. $\frac{2}{x^2 - 2x - 8}$
6. $\frac{x - 9}{x^2 - 3x - 18}$
7. $\frac{x^2}{x^3 - 16x}$
8. $\frac{x^2 - 3x}{x^3 - 3x^2 - 4x}$

9–18. Simple linear factors *Evaluate the following integrals.*

9. $\int \frac{dx}{(x-1)(x+2)}$
10. $\int \frac{8}{(x-2)(x+6)}\,dx$
11. $\int \frac{3}{x^2 - 1}\,dx$
12. $\int \frac{dt}{t^2 - 9}$
13. $\int \frac{2}{x^2 - x - 6}\,dx$
14. $\int \frac{3}{x^3 - x^2 - 12x}\,dx$
15. $\int \frac{dx}{x^2 - 2x - 24}$
16. $\int \frac{y + 1}{y^3 + 3y^2 - 18y}\,dy$
17. $\int \frac{1}{x^4 - 10x^2 + 9}\,dx$
18. $\int \frac{2}{x^2 - 4x - 32}\,dx$

19–25. Repeated linear factors *Evaluate the following integrals.*

19. $\int \frac{3}{x^3 - 9x^2}\,dx$
20. $\int \frac{x}{(x-6)(x+2)^2}\,dx$
21. $\int \frac{x}{(x+3)^2}\,dx$
22. $\int \frac{dx}{x^3 - 2x^2 - 4x + 8}$

23. $\int \frac{2}{x^3 + x^2}\,dx$

24. $\int \frac{2}{t^3(t + 1)}\,dt$

25. $\int \frac{x - 5}{x^2(x + 1)}\,dx$

26–29. Setting up partial fraction decompositions *Give the appropriate form of the partial fraction decomposition for the following functions.*

26. $\frac{2}{x(x^2 - 6x + 9)}$

27. $\frac{20x}{(x - 1)^2(x^2 + 1)}$

28. $\frac{x^2}{x^3(x^2 + 1)}$

29. $\frac{2x^2 + 3}{(x^2 - 8x + 16)(x^2 + 3x + 4)}$

30–36. Simple irreducible quadratic factors *Evaluate the following integrals.*

30. $\int \frac{x^2 + 2}{x(x^2 + 5x + 8)}\,dx$

31. $\int \frac{2}{(x - 4)(x^2 + 2x + 6)}\,dx$

32. $\int \frac{z + 1}{z(z^2 + 4)}\,dz$

33. $\int \frac{x^2}{(x - 1)(x^2 + 4x + 5)}\,dx$

34. $\int \frac{2x + 1}{x^2 + 4}\,dx$

35. $\int \frac{x^2}{x^3 - x^2 + 4x - 4}\,dx$

36. $\int \frac{1}{(y^2 + 1)(y^2 + 2)}\,dy$

Further Explorations

37. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. To evaluate $\int \frac{4x^6}{x^4 + 3x^2}\,dx$, the first step is to find the partial fraction decomposition of the integrand.
   b. The easiest way to evaluate $\int \frac{6x + 1}{3x^2 + x}\,dx$ is with a partial fraction decomposition of the integrand.
   c. The rational function $f(x) = \frac{1}{x^2 - 13x + 42}$ has an irreducible quadratic denominator.
   d. The rational function $f(x) = \frac{1}{x^2 - 13x + 43}$ has an irreducible quadratic denominator.

38–41. Areas of regions *Find the area of the following regions. In each case, graph the relevant functions and show the region in question.*

38. The region bounded by the curve $y = x/(1 + x)$, the x-axis, and the line $x = 4$.

39. The region bounded by the curve $y = 10/(x^2 - 2x - 24)$, the x-axis, and the lines $x = -2$ and $x = 2$.

40. The region bounded by the curves $y = 1/x$, $y = x/(3x + 4)$, and the line $x = 10$.

41. The region bounded entirely by the curve $y = \frac{x^2 - 4x - 4}{x^2 - 4x - 5}$ and the x-axis.

42–47. Volumes of solids *Find the volume of the following solids.*

42. The region bounded by $y = 1/(x + 1)$, $y = 0$, $x = 0$, and $x = 2$ is revolved about the y-axis.

43. The region bounded by $y = x/(x + 1)$, the x-axis, and $x = 4$ is revolved about the x-axis.

44. The region bounded by $y = (1 - x^2)^{-1/2}$ and $y = 4$ is revolved about the x-axis.

45. The region bounded by $y = \frac{1}{\sqrt{x(3 - x)}}$, $y = 0$, $x = 1$, and $x = 2$ is revolved about the x-axis.

46. The region bounded by $y = \frac{1}{\sqrt{4 - x^2}}$, $y = 0$, $x = -1$, and $x = 1$ is revolved about the x-axis.

47. The region bounded by $y = 1/(x + 2)$, $y = 0$, $x = 0$, and $x = 3$ is revolved about the line $x = -1$.

48. **What's wrong?** Explain why the coefficients A and B cannot be found if we set
$$\frac{x^2}{(x - 4)(x + 5)} = \frac{A}{x - 4} + \frac{B}{x + 5}.$$

49–59. Preliminary steps *The following integrals require a preliminary step such as long division or a change of variables before using partial fractions. Evaluate these integrals.*

49. $\int \frac{dx}{1 + e^x}$

50. $\int \frac{x^4 + 1}{x^3 + 9x}\,dx$

51. $\int \frac{3x^2 + 4x - 6}{x^2 - 3x + 2}\,dx$

52. $\int \frac{2x^3 + x^2 - 6x + 7}{x^2 + x - 6}\,dx$

53. $\int \frac{dt}{2 + e^{-t}}$

54. $\int \frac{dx}{e^x + e^{2x}}$

55. $\int \frac{\sec\theta}{1 + \sin\theta}\,d\theta$

56. $\int \sqrt{e^x + 1}\,dx$

57. $\int \frac{e^x}{(e^x - 1)(e^x + 2)}\,dx$

58. $\int \frac{\cos x}{(\sin^3 x - 4\sin x)}\,dx$

59. $\int \frac{dx}{(e^x + e^{-x})^2}$

60–65. Fractional powers *Use the indicated substitution to convert the given integral to an integral of a rational function. Evaluate the resulting integral.*

60. $\int \frac{dx}{x - \sqrt[3]{x}}$; $x = u^3$

61. $\int \frac{dx}{\sqrt[4]{x + 2} + 1}$; $x + 2 = u^4$

62. $\int \frac{dx}{x\sqrt{1 + 2x}}$; $1 + 2x = u^2$

63. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$; $x = u^6$

64. $\int \frac{dx}{x - \sqrt[4]{x}}$; $x = u^4$

65. $\int \frac{dx}{\sqrt{1 + \sqrt{x}}}$; $x = (u^2 - 1)^2$

66. **Arc length of the natural logarithm** Consider the curve $y = \ln x$.
   a. Find the length of the curve from $x = 1$ to $x = a$ and call it $L(a)$. (*Hint:* The change of variables $u = \sqrt{x^2 + 1}$ allows evaluation by partial fractions.)
   b. Graph $L(a)$.
   c. As a increases, $L(a)$ increases as what power of a?

67–70. Repeated quadratic factors *Refer to the summary box on p. 483 and evaluate the following integrals.*

67. $\int \frac{2}{x(x^2 + 1)^2}\,dx$

68. $\int \frac{dx}{(x + 1)(x^2 + 2x + 2)^2}$

69. $\int \frac{x}{(x - 1)(x^2 + 2x + 2)^2}\,dx$

70. $\int \frac{x^3 + 1}{x(x^2 + x + 1)^2}\,dx$

71. **Two methods** Evaluate $\int \frac{dx}{x^2 - 1}$ for $x > 1$ in two ways: using partial fractions and a trigonometric substitution. Reconcile your two answers.

72–78. Rational functions of trigonometric functions *An integrand with trigonometric functions in the numerator and denominator can often be converted to a rational integrand using the substitution $u = \tan(x/2)$ or $x = 2\tan^{-1} u$. The following relations are used in making this change of variables.*

$$A:\ dx = \frac{2}{1 + u^2}\,du \qquad B:\ \sin x = \frac{2u}{1 + u^2} \qquad C:\ \cos x = \frac{1 - u^2}{1 + u^2}$$

72. Verify relation A by differentiating $x = 2\tan^{-1} u$. Verify relations B and C using a right-triangle diagram and the double-angle formulas
$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \text{ and } \cos x = 2\cos^2\left(\frac{x}{2}\right) - 1.$$

73. Evaluate $\int \frac{dx}{1 + \sin x}$.

74. Evaluate $\int \frac{dx}{2 + \cos x}$.

75. Evaluate $\int \frac{dx}{1 - \cos x}$.

76. Evaluate $\int \frac{dx}{1 + \sin x + \cos x}$.

77. Evaluate $\int \frac{d\theta}{\cos\theta - \sin\theta}$.

78. Evaluate $\int \sec t\,dt$.

Applications

79. **Three start-ups** Three cars, A, B, and C, start from rest and accelerate along a line according to the following velocity functions:
$$v_A(t) = \frac{88t}{t + 1} \qquad v_B(t) = \frac{88t^2}{(t + 1)^2} \qquad v_C(t) = \frac{88t^2}{t^2 + 1}$$
   a. After $t = 1$ s, which car has traveled farthest?
   b. After $t = 5$ s, which car has traveled farthest?
   c. Find the position functions for the three cars assuming that all cars start at the origin.
   d. Which car ultimately gains the lead and remains in front?

80. **Skydiving** A skydiver has a downward velocity given by
$$v(t) = V\left(\frac{1 - e^{-2gt/V}}{1 + e^{-2gt/V}}\right),$$
where $t = 0$ is the instant the skydiver starts falling, $g \approx 9.8\ \text{m/s}^2$ is the acceleration due to gravity, and V is the terminal velocity of the skydiver.
   a. Evaluate $v(0)$ and $\lim_{t\to\infty} v(t)$ and interpret these results.
   b. Graph the velocity function.
   c. Verify by integration that the position function is given by
$$s(t) = Vt + \frac{V^2}{g}\ln\left(\frac{1 + e^{-2gt/V}}{2}\right)$$
   where $s'(t) = v(t)$ and $s(0) = 0$.
   d. Graph the position function.

   (See the Guided Projects for more details on free fall and terminal velocity.)

Additional Exercises

81. $\pi < \frac{22}{7}$ One of the earliest approximations to π is $\frac{22}{7}$. Verify that $0 < \int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\,dx = \frac{22}{7} - \pi$. Why can you conclude that $\pi < \frac{22}{7}$?

82. **Challenge** Show that with the change of variables $u = \sqrt{\tan x}$, the integral $\int \sqrt{\tan x}\,dx$ can be converted to an integral amenable to partial fractions. Evaluate $\int_0^{\pi/4} \sqrt{\tan x}\,dx$.

QUICK CHECK ANSWERS

1. $\ln|x - 2| + 2\ln|x + 4| = \ln|(x - 2)(x + 4)^2|$
2. $A/(x - 1) + B/(x + 5) + C/(x - 10)$
3. $A/x + B/x^2 + C/(x - 3) + D/(x - 3)^2 + E/(x - 1)$ ◀